Class 3, given on Jan 8, 2010, for Math 13, Winter 2010

1. Vector-valued functions

A vector-valued function is a function on $\mathbb{R}$ which takes values in some $\mathbb{R}^n$. We usually write the independent variable as t, and then write $\mathbf{r}(t) = \langle r_1(t), \ldots, r_n(t)$ for the components of the vector-valued function $\mathbf{r}$. Sometimes we may write $\langle x(t), y(t)\rangle$ or $\langle x(t), y(t), z(t)\rangle$ in the cases where the function takes values in $\mathbb{R}^2$ or $\mathbb{R}^3$.

The graph of a vector-valued function will be a curve in $\mathbb{R}^n$.

Examples.

- The graph of $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$ is the unit circle. Notice that a function like $\mathbf{r}(t) = \langle \cos 2t, \sin 2t\rangle$ has the same graph, but is a different function, so a graph does not uniquely determine a vector-valued function.
- Suppose we are given two points $\mathbf{r_0}$ and $\mathbf{r_1}$. What is a vector-valued function whose graph is a line segment connecting these two points? If we form the vector $\mathbf{r_1} - \mathbf{r_0}$, then one quickly sees that

$$\mathbf{r}(t) = \mathbf{r_0} + t\mathbf{r_1} - \mathbf{r_0} = (1-t)\mathbf{r_0} + t\mathbf{r_1}, 0 \le t \le 1$$

  has a graph which is the line segment connecting $\mathbf{r_0}$ with $\mathbf{r_1}$. This calculation will be handy in the latter half of this class.
- The graph of $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$ is a helix, whose loops are separated by a distance of 2π. Indeed, as t increases, the x, y coordinates wind around in a circle, but the z coordinate increases at a constant, linear rate.
- What is the graph of $\mathbf{r}(t) = \langle \cos t, \sin t, \sin 6t\rangle$? This evidently looks like the graph of a sine function wrapped around in a circle. How many periods (ie, how often does the sine function repeat) in this circular loop?

We can take derivatives and integrals of vector-valued functions by taking usual derivatives and integrals component by component. The derivative of a vector-valued function $\mathbf{r}(t)$, $\mathbf{r}'(t)$, is useful in a variety of contexts.

For example, if $\mathbf{r}(t)$ describes the motion of a particle at time t, then $\mathbf{r}'(t)$ is the velocity vector for that particle at time t. The speed of the particle is given by $|\mathbf{r}'(t)|$ – the length of the velocity vector. Similarly, the acceleration of the particle is given by $\mathbf{r}''(t)$. We may also have occasion to use the *unit tangent vector* of $\mathbf{r}(t)$, which is the vector-valued function $\mathbf{T}(t)$ of length 1 which points in the same direciton as $\mathbf{r}'(t)$:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

The derivative $\mathbf{r}'(t)$ also allows us to calculate the tangent line to the graph of $\mathbf{r}(t)$ at various points on the curve, because this derivative is a direction vector for the tangent line.

Example. Find parametric equations for the tangent line to $\mathbf{r}(t) = \langle t^2, 2^t, \log t$ at the point $(1, 2, 0)$. The derivative of this function is $\mathbf{r}'(t) = \langle 2t, 2^t(\log 2), 1/t\rangle$. The point $(1, 2, 0)$ is equal to $\mathbf{r}(1)$. Therefore, a direction vector for this tangent line is given by $\mathbf{r}'(1) = \langle 2, 2\log 2, 1\rangle$. In particular, parametric equations for the tangent line are given by $\ell(t) = \langle 1 + 2t, 2 + 2\log 2t, t\rangle$.

2. Arc length

Consider the graph of a vector-valued function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ when $a \leq t \leq b$. This is a segment of a curve, and so probably has a length. If each point of this curve is touched exactly once by $\mathbf{r}(t)$, except possibly at a finite number of points, we define the arc length of such a curve to be the value of the integral

$$\int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt.$$

This integral has an obvious generalization to vector-valued functions that take values in other $\mathbb{R}^n$.

Example. Use the arc length integral to calculate the circumference of a unit circle. We can choose any parameterization of the unit circle; for example, the usual parameterization $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$ works. We also need to decide the range t should fall in to ensure that that $\mathbf{r}(t)$ traverses the circle exactly once. In this case, the choice $0 \leq t \leq 2\pi$ works. The arc length integral is then

$$\int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt = \int_0^{2\pi} 1\, dt = 2\pi.$$

This is what we expect the answer to be from geometry!

The truth of the matter is that one should not generally expect to be able to evaluate arc length integrals exactly, because integrating a quantity under a square root sign is usually a very difficult, if not impossible, question. For example, even in the case of ellipses, it is in general impossible to evaluate the integrals that appear when calculating the arc length of ellipses exactly. (Such integrals are known as 'elliptic integrals' and their study was one of the most fruitful branches of mathematics in the early 19th century.)

3. Functions of several variables

This class will be concerned with functions of several variables. Usually, their domain will be subsets of $\mathbb{R}^2$ or $\mathbb{R}^3$, and we often write $f(x, y)$ or $f(x, y, z)$ for these functions. For now, these functions will take values in $\mathbb{R}$, but halfway through the class we will encounter *vector fields*, which are functions $f : \mathbb{R}^2 \to \mathbb{R}^2$, or $f : \mathbb{R}^3 \to \mathbb{R}^3$.

The graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$ is a surface in three-dimensional space. In general, it is hard to graph these functions by hand (think back to the fact that it is not easy to even graph functions of one variable by hand), so we sometimes use other tools to help us visualize these functions. An obvious tool is to use a graphing calculator or computer package, but it is also important to gain a geometric intuition for these functions without machine assistance. (Incidentally, Dartmouth offers free downloads of Maple, a computer algebra system.)

One way of visualizing graphs $z = f(x, y)$ is by means of plotting *level curves*. A level curve of $f(x, y)$ is just a set of points in the xy plane which satisfy an equation $f(x, y) = C$, where C is some constant. If you have ever seen a topographic map (common in hiking), the curves on these maps are level curves. Similarly, a map of temperature with isotherms (curves where temperature is constant) is another example of a map with level curves on it.

Examples.

- Consider the level curves of the function $z^2 = x^2 + y^2$. We are plotting curves $C = \sqrt{x^2 + y^2}$, which are circles with radius C. In particular, these level curves are evenly spaced, in the sense that as C increments by some fixed amount, the spacing between the level curves also increment by some fixed amount. A plot of this graph shows that this function determines a cone (actually, two cones).

- Consider the level curves of $z = x^2 + y^2$. This time, our level curves are the set of points satisfying $C = x^2 + y^2$, which are circles with radius $\sqrt{C}$. These level curves become more and more tightly bunched as C grows, which reflects the fact that the value of this function changes more rapidly as we get further from the origin.

4. Partial derivatives

What is a 'derivative' of a function of several variables? It turns out that the best answer requires some knowledge of linear algebra, so we do not worry too much about this question in this class. However, we do need to know about various candidates for derivatives of these functions.

The most straightforward of these ideas is that of a *partial derivative*. Given a function $f(x, y)$, the partial derivative of $f(x, y)$ with respect to x is calculated by treating y as a constant and then differentiating the function with respect to x. Similarly, we can calculate the partial derivative of $f(x, y)$ with respect to y. The partial derivatives are actually defined using a limit, similar to how derivatives of functions of a single variable are defined. We sometimes write f_x or f_y for partial derivatives, and sometimes use the following analogue of Leibniz notation:

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}.$$

Example. Let $f(x, y) = x^2y + e^y \cos x$. Then $f_x = 2xy - e^y \sin x$, $f_y = x^2 + e^y \cos x$.

We can also define higher order partial derivatives, such as f_{xx}, by taking a partial derivative of f_x. Partial derivatives like f_{xy}, where we take derivatives with respect to different variables, are known as *mixed partial derivatives*. When we write f_{xy}, we mean the partial derivative of f_x with respect to y. However, in the Leibniz notation, the ordering of the indices is reversed:

$$f_{xy} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x}.$$

Why do we make a fuss about the ordering of the indices? Because in general, it is not always true that $f_{xy} = f_{yx}$! However, a theorem known as Clairaut's Theorem (more commonly known as Young's Theorem) guarantees that $f_{xy} = f_{yx}$ in virtually every situation we will encounter.

Theorem. (Clairaut's Theorem, Young's Theorem) Let $f(x, y)$ be defined in a disc D containing the point (a, b), such that f_{xy}, f_{yx} are both continuous on D. Then $f_{xy}(a, b) = f_{yx}(a, b)$.

For a function whose mixed partials are not equal, consult Problem 95 in Chapter 15.3 of the text.

What is the geometric interpretation of partial derivatives? Just like how the derivative of a single-variable function is the slope of the graph of a function, the value of a partial derivative tells us the rate of change of a function as we move in either the x or y direction. Another way of thinking about this is that the value of $f_x(a, b)$ tells us the rate of change of the curve on the surface $z = f(x, y)$ obtained by looking only at points with $y = b$, and letting x vary, at the point $x = a$.

Example. Let $f(x,y) = x^2 + 2y^2$. At the point $(1,2)$, the partial derivatives of f are equal to $f_x(1,2) = 2$, $f_y(1,2) = 8$. The cross-section of the graph obtained by letting $y = 2$ has points $(x, 2, x^2 + 8)$. The 'slope' of this curve at $x = 1$ is then given by $2(1) = 2$. Similarly, if we take the cross-section of this graph with $x = 1$, then the 'slope' of the resulting curve, which is the set of points $(1, y, 1 + 2y^2)$ at $y = 2$ is given by $4(2) = 8$.

All of this discussion extends in the obvious way to functions of more than two variables. For example, when taking the partial derivative of $f(x,y,z)$ with respect to x, we treat both y, z as constant, and take a derivative with respect to x.

Example. Let $f(x,y,z) = 2xy + yz^2 + z\log x$. Then $f_x(x,y,z) = 2y + z/x$, $f_y(x,y,z) = 2x + z^2$, $f_z(x,y,z) = 2yz + \log x$.

In this class you will be taking lots of partial derivatives, so make sure you become accurate and fairly quick at it!